Proofs for theorems in:

"Tester versus Bug: A Generic Framework for Model-Based Testing via Games"

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Proof Theorem 1

Let $\mathscr{A} = (Q, L_I, L_O, T, q_0)$ be an SA, and $G^{IE}_{\mathscr{A}}, G^{IF}_{\mathscr{A}}, G^{OE}_{\mathscr{A}}, G^{ND}_{\mathscr{A}}$ the underlying game arenas of $\mathscr{A}$ for the input-eager, input-fair, output-eager, and nondeterministic test assumption, respectively. Let $R \subseteq Q$ be a reachability goal. We need to prove:

1. $G^{IE}_{\mathscr{A}}$ is winning for player 1 w.r.t. R if and only if $G^{IF}_{\mathscr{A}}$ is winning for player 1 w.r.t. R, and
2. $G^{OE}_{\mathscr{A}}$ is winning for player 1 w.r.t R if and only if $G^{ND}_{\mathscr{A}}$ is winning for player 1 w.r.t. R.

Direction 1. $\Longleftarrow$

We are given game $G^{IF}_{\mathscr{A}}$, which is winning for player 1 w.r.t. R. Hence there exists a winning player 1 strategy $\sigma_1 \in \Sigma_1(G^{IF}_{\mathscr{A}})$. According to Definition 12, σ_1 is winning on all input-fair plays of $G^{IF}_{\mathscr{A}}$. We have that $\Pi^{pref}(G^{IE}_{\mathscr{A}})$ contains a subset of these fair plays (Definition 11), since the *Moves* function for the input eager assumption (Definition 9) always returns the set of the single next state after an input transition, if player one provides an input $a \in L_I$, and this is input-fair. Consequently, we can restrict the domain of σ_1 to obtain a winning strategy for $G^{IE}_{\mathscr{A}}$.

Direction 1. $\Longrightarrow$

We are given game $G^{IE}_{\mathscr{A}}$, which is winning for player 1 w.r.t. R. Hence there exists a winning player 1 strategy $\sigma_1 \in \Sigma_1(G^{IE}_{\mathscr{A}})$. Since winning strategies in reachability games need only to be positional [1], we can use σ_1 to construct a winning strategy $G^{IF}_{\mathscr{A}}$ by just considering the last state of a play. Hence, we define $\sigma_1' \in \Sigma_1(G^{IF}_{\mathscr{A}})$ as $\sigma_1'(\pi) = \sigma(\pi')$ for some $\pi' \in \Pi^{pref}(G^{IE}_{\mathscr{A}})$ with $(\pi')^q_{|\pi'|-1} = \pi^q_{|\pi|-1}$. This way we defined a winning strategy on the biger domain of play prefixes from game $G^{IF}_{\mathscr{A}}$.

Direction 2. $\Longrightarrow$

We are given game $G^{OE}_{\mathscr{A}}$, which is winning for player 1 w.r.t. R. Hence there exists a winning player 1 strategy $\sigma_1 \in \Sigma_1(G^{OE}_{\mathscr{A}})$. Since winning strategies in reachability games need only to be positional [1], we can, again, use σ_1 to construct a winning strategy $G^{ND}_{\mathscr{A}}$ by just considering the last state of a play: define $\sigma_1' \in \Sigma_1(G^{ND}_{\mathscr{A}})$ as $\sigma_1'(\pi) = \sigma(\pi')$ for some $\pi' \in \Pi^{pref}(G^{OE}_{\mathscr{A}})$ with $(\pi')^q_{|\pi'|-1} = \pi^q_{|\pi|-1}$.

Direction 2. $\Longleftarrow$

We are given game $G^{ND}_{\mathscr{A}}$, which is winning for player 1 w.r.t. R. Hence there exists a winning player 1 strategy $\sigma_1 \in \Sigma_1(G^{ND}_{\mathscr{A}})$. If $\sigma_1(\pi) \in \mathrm{Act}_1$ for some play prefix π, but $\Gamma_2(\pi^q_{|\pi|-1}) \neq \{\delta\}$ (i.e. the last state of π is mixed), then σ_1 can only be winning if a move for any player 2 action (instead of $\sigma_1(\pi)$) is also a winning move, i.e. the play in which this player 2 action is taken is also winning, because the reached state for taking a player 2 action is included in the result of the *Moves* function according to Definition 9. Consequently, we can just restrict the domain of play prefixes of σ_1' to the play prefixes of $G^{OE}_{\mathscr{A}}$ to obtain a winning player 1 strategy in $G^{OE}_{\mathscr{A}}$.

Proof Theorem 2

Let $\mathscr{A}$ be an SA, and $G_{\mathscr{A}}$ the underlying game arena. Let $\sigma_1 \in \Sigma_1(G_{\mathscr{A}})$ be a player 1, trace-based, finite strategy in $G_{\mathscr{A}}$. We define a trace set $T_{\sigma_1} = \{trace(\pi) \mid \pi \in Pref(Outc(\sigma_1)) \wedge \pi^a_{|\pi|-1} \neq \text{stop?}\}$. To prove: T_{σ_1} characterizes a unique test case.

Let $\pi \in Pref(Outc(\sigma_1))$ such that $\pi^a_{|\pi|-1} \neq$ stop?. From Definition 8 we know that a play prefix having $\pi^a_j =$ stop? for some $j \in \mathbb{N}$ has $\pi^a_k =$ stop? for any $j \leq k \leq |\pi|-1$, i.e. all player 1 actions of a play after action stop? are also stop?. Consequently, $\pi^a_{|\pi|-1} \neq$ stop? means that π does not contain any player 1 action stop?, so all play prefixes are 'cut off' at their stop? action. Therefore, $trace(\pi) \in L^*$. Also, the requirement $\pi^a_{|\pi|-1} \neq$ stop? does not interfere with the set $Pref(Outc(\sigma_1))$ being prefix closed. This means that the resulting trace set is also prefix closed. As it is finite (because of cutting of at stop? actions, and the finiteness of $\mathscr{A}$), we can construct a test case the following way:

1. Let $Q \subset T_{\sigma_1}$ be the set of traces which all are a proper prefix of some trace in T_{σ_1}, i.e. they are not the longest prefix. Then each element of Q identifies a state of the test case, and transitions are defined according to their prefix relation, i.e. $\rho = \rho'\mu$ for some $\mu \in L$ implies $T(\rho', \mu) = \rho$.
2. The initial state of the test case is the empty trace ε.
3. Let $P \subset T_{\sigma_1}$ be the set of longest prefixes. For each $\rho\mu \in P$ for some $\mu \in L$, set $T(\rho, \mu) = Pass$. Define state *Pass* conforming to Definition 7.
4. For all states $\rho \in Q$, set $T(\rho, x) = Fail$ for all $x \in L_O$ with $\rho x \notin T_{\sigma_1}$. Define state *Fail* conforming to Definition 7.

We now check that this definition conforms to Definition 7.

1. The states *Pass* and *Fail* are defined according to Definition 7.
2. The other states, and their transitions are defined according to the prefix relation. This ensures that there are no cycles except those in *Pass* and *Fail*.
3. As σ_1 returns one player 1 action, states of the test case cannot have more than one input transition. Note that the input θ never occurs in the traces by Definition 9, it only enables δ to be present. By the addition of an output transition for all outputs (point 4) not enabled after a state, we obtain the requirement $|in(q)| = 0 \wedge out(q) = L^{\delta}_O) \vee (out(q) = L_O \wedge |in(q)| = 1)$.
4. That traces of the test case leading to *Pass* are traces of $\mathscr{A}$, while traces of *Fail* are not, follows from the construction of the game $G_{\mathscr{A}}$ from the SA $\mathscr{A}$ (Definition 8), and the fact that all longest traces of T_{σ_1} have become *Pass* states in our construction, while transition to *Fail* states have to be added additionally (point 4).

Proof Theorem 3

Let $\mathscr{T}$ be a test case for SA $\mathscr{A}$. To prove:

1. The strategy $\sigma_{\mathscr{T}}$ is a Player 1 strategy in $G_{\mathscr{A}}$.
2. $\sigma_{\mathscr{T}}$ is finite and trace-based.
3. If $\sigma_{\mathscr{T}} = \sigma_{\mathscr{T}'}$ then $\mathscr{T} = \mathscr{T}'$.

Statement 1 Cearly, Definition 15 defines $\sigma_{\mathscr{T}}$ as a function from $\Pi^{pref}(G_{\mathscr{A}})$ to $\mathrm{Act}_1 = L_I \cup \{\theta, \mathrm{stop?}\}$. We also need $\sigma_i(\pi) \in \Gamma_1(\pi^q_{|\pi|-1})$ for any $\pi \in \Pi^{pref}(G_{\mathscr{A}})$. As stop? and θ are enabled in any game state, we need to check this for $\sigma_i(\pi) \in L_I$. In this case Definition 15 requires that $\mathscr{T}$ *after* $(trace(\pi)) = \{q\} \wedge in(\{\}q\} = \{a\}$, even if a is not enabled after $trace(\pi)$ in specification $\mathscr{A}$. Therefore this case additionally should require $\exists q' \in Q : \mathscr{A}$ *after* $(trace(\pi)) = \{q'\} \wedge a \in in(\{\}q'\}$. If this does not hold stop? should be chosen, as the test case does not tell what to test instead of a (note that θ is not the right choice here because $\sigma_{\mathscr{T}}$ only chooses θ for plays that are not contained in $Outc(\sigma_{\mathscr{T}})$).

Statement 2 We have by Definition 7 that a test case is acyclic (except in its *Pass* and *Fail* state). Because we encode state *Pass*, and inputs not enabled in the specification, with a stop? in strategy $\sigma_{\mathscr{T}}$, we then have that $\sigma_{\mathscr{T}}$ is finite. Note here that a *Fail* state is never reached by a trace of a play, as the trace in a test case to a *Fail* state is not present in the specification. As $\sigma_{\mathscr{T}}$ only uses the trace of a play prefix to determine the returned action, $\sigma_{\mathscr{T}}$ is trace-based by construction.

Statement 3 This follows from the fact that a test case is defined by its trace-prefixes (we used this property in Theorem 2 to construct a test case).

Proof Theorem 4

This follows from Theorem 2 and Theorem 3, since the constructions from Definition 14 and Definition 15 are defined to be each others inverse. A test case is charachterized by its trace-prefixes. This is exactly what is captured by finite, trace-based strategies.

Proof Theorem 5

Let $\mathscr{A}$ and $\mathscr{B}$ be SAs over the same label sets and assume that $\mathscr{A}$ is input-enabled. Let $G_{\mathscr{A}}$ and $G_{\mathscr{B}}$ be their respective underlying game arenas for the nondeterministic test assumption. To prove:

$$\mathscr{A} \ ioco \ \mathscr{B} \iff G_{\mathscr{A}} \sqsubseteq_2 G_{\mathscr{B}}$$

Direction $\implies$

We are given $\mathscr{A}$ *ioco* $\mathscr{B}$, and need to prove:

σ_1^A does not cheat on σ_1^B, and

$$\{trace(\pi) \mid \pi \in Pref(Outc(\sigma_1^A, \sigma_2^A))\} \subseteq \{trace(\pi) \mid \pi \in Pref(Outc(\sigma_1^B, \sigma_2^B))\} \quad \text{(P)}$$

for strategies $\sigma_2^A \in \Sigma_2(G_A)$, $\sigma_2^B \in \Sigma_2(G_B)$, $\sigma_1^B \in \Sigma_1(G_B)$, and $\sigma_1^A \in \Sigma_1(G_A)$. In the deduction tree of Figure 1 we abreviate this formula to $P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A)$. The notations $\forall I$ and $\exists I$ refer to the forall introduction and exists introduction rule, respectively. The application of these rules in this deduction tree, requires that we choose a specific strategy $\sigma_2^B \in \Sigma_2(G_B)$, given an arbitrary strategy $\sigma_2^A \in \Sigma_2(G_{\mathscr{A}})$, and that we choose a specific strategy $\sigma_1^A \in \Sigma_1(G_A)$, given σ_2^A, σ_2^B, and arbitrary strategy $\sigma_1^B \in \Sigma_1(G_{\mathscr{B}})$. Choosing these strategies will enable us to proof the formula $P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A)$ as required by the 'To Prove' rule.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathscr{A}\ ioco\ \mathscr{B} \vdash P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A)}\ \text{To Prove}}{\mathscr{A}\ ioco\ \mathscr{B} \vdash \exists \sigma_1^A \in \Sigma_1(G_A) : P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A)}\ \exists I}{\mathscr{A}\ ioco\ \mathscr{B} \vdash \forall \sigma_1^B \in \Sigma_1(G_B), \exists \sigma_1^A \in \Sigma_1(G_A) : P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A)}\ \forall I}{\exists \sigma_2^B \in \Sigma_2(G_B), \forall \sigma_1^B \in \Sigma_1(G_B), \exists \sigma_1^A \in \Sigma_1(G_A) : P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A)}\ \exists I}{\mathscr{A}\ ioco\ \mathscr{B} \vdash \forall \sigma_2^A \in \Sigma_2(G_A), \exists \sigma_2^B \in \Sigma_2(G_B), \forall \sigma_1^B \in \Sigma_1(G_B), \exists \sigma_1^A \in \Sigma_1(G_A) : P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A)}\ \forall I
$$

Figure 1: Deduction tree for case $\Longrightarrow$.

In the definition of these two strategies, we translate play prefixes from one game to the other via their action decision sequences. For this we use the functions $t_{\mathscr{A}} : \Pi^{pref}(G_{\mathscr{A}}) \rightharpoonup \Pi^{pref}(G_{\mathscr{B}})$ and $t_{\mathscr{B}} : \Pi^{pref}(G_{\mathscr{B}}) \rightharpoonup \Pi^{pref}(G_{\mathscr{A}})$. We define $t_{\mathscr{A}}(\pi) = \pi'$ if there exists a $\pi' \in \Pi^{pref}(G_{\mathscr{B}})$ with $actions(\pi) = actions(\pi')$, and undefined otherwise. The inverse $t_{\mathscr{B}}$ is defined similarly.

We choose the two strategies σ_2^B and σ_1^A according to the definition as given below. Note that σ_1^A is non-cheating, as it chooses the same input for all play prefixes in $G_{\mathscr{A}}$ that have a play prefix in $G_{\mathscr{B}}$ with the same action decision sequence. As $\mathscr{A}$ is input-enabled, this is well-defined.

$$
\sigma_2^B(\pi) = \begin{cases} \sigma_2^A(t_{\mathscr{B}}(\pi)) & \text{if } t_{\mathscr{B}}(\pi) \text{ defined and } \sigma_2^A(t_{\mathscr{B}}(\pi)) \in \Gamma_2^{\mathscr{B}}(\pi^q_{|\pi|-1}) \\ \text{an arbitrary } x \in \Gamma_2^{\mathscr{B}}(\pi^q_{|\pi|-1}) & \text{otherwise} \end{cases}
$$

$$
\sigma_1^A(\pi) = \begin{cases} \sigma_1^B(t_{\mathscr{A}}(\pi)) & \text{if } t_{\mathscr{A}}(\pi) \text{ defined} \\ \theta & \text{otherwise} \end{cases}
$$

We now will prove the 'To Prove' rule of the deduction tree of Figure 1. Let $\pi \in Pref(Outc(\sigma_1^A, \sigma_2^A))$. Because σ_1^A does not cheat on σ_1^B, it remains to prove that $trace(\pi) \in \{trace(\pi') \mid \pi' \in Outc(\sigma_1^B, \sigma_2^B)\}$. We use induction on the length of π to prove this, and also $t_{\mathscr{A}}(\pi)$ defined and $t_{\mathscr{A}}(\pi) \in Pref(Outc(\sigma_1^B, \sigma_2^B))$.

If $|\pi| = 1$, i.e. $\pi = q_0^{\mathscr{A}}$, then $trace(\pi) = \varepsilon = trace(t_{\mathscr{A}}(\pi)) = trace(q_0^{\mathscr{B}}) \in Pref(Outc(\sigma_1^B, \sigma_2^B))$. We then obtain $trace(\pi) \in \{trace(\pi') \mid \pi' \in Pref(Outc(\sigma_1^B, \sigma_2^B)\})$, as required.

Now suppose that $|\pi| = n+1$ for some $n \in \mathbb{N} \setminus \{0\}$. Take decomposition $\pi = \psi\langle a, x\rangle(q, j)$. We first prove that $t_{\mathscr{A}}(\pi)$ is defined and $t_{\mathscr{A}}(\pi) \in Pref(Outc(\sigma_1^B, \sigma_2^B))$.

From $x \in \Gamma_2^{\mathscr{A}}(\psi^q_{|\psi|-1})$ we know by Definition 8 that $x \in out(\mathscr{A}\ after\ (trace(\psi)))$. From $\mathscr{A}\ ioco\ \mathscr{B}$ we then obtain $x \in out(\mathscr{B}\ after\ (trace(\psi)))$. Since $t_{\mathscr{A}}(\psi)$ defined by the induction hypothesis, we have $x \in \Gamma_2^{\mathscr{B}}(t_{\mathscr{A}}(\psi)^q_{|\psi|-1})$. As SA $\mathscr{A}$ is input enabled, we then have that $t_{\mathscr{A}}(\pi)$ defined. Now note that $\sigma_1^A(\psi) = \sigma_1^B(t_{\mathscr{A}}(\psi)) = a$, i.e. σ_1^A returns an input as choosen by σ_1^B, according to its definition. Furthermore, it follows from the definition of σ_2^B that $\sigma_2^B(t_{\mathscr{A}}(\psi)) = \sigma_2^A(t_{\mathscr{B}}(t_{\mathscr{A}}(\psi))) = \sigma_2^A(\psi) = x$, i.e. σ_2^A chooses the output that σ_2^B chooses. With the induction hypothesis, we then obtain $t_{\mathscr{A}}(\pi) \in Pref(Outc(\sigma_1^B, \sigma_2^B))$. We use this to prove that $trace(\pi) \in \{trace(\pi') \mid \pi' \in Outc(\sigma_1^B, \sigma_2^B)\}$.

If $j=2$, then $trace(\pi) = trace(\psi)x$. We have $x \in \Gamma_2^{\mathscr{B}}(t_{\mathscr{A}}(\psi)^q_{|\psi|-1})$ and $\sigma_2^A(\psi) = \sigma_2^B(t_{\mathscr{A}}(\psi)) = x$ (for the reasons explained above). By the nondeterministic test assumption we then have $trace(\psi)x = trace(\pi) \in \{trace(\pi') \mid \pi' \in Pref(Outc(\sigma_1^B, \sigma_2^B))\}$.

If $j=1$, then $trace(\pi) = trace(\psi)a$. Hence, $a \in \Gamma_1^{\mathscr{A}}(\psi^q_{|\psi|-1})$, and $\sigma_1^A(\psi) = \sigma_1^B(t_{\mathscr{A}}(\psi)) = a$ (for the reasons explained above). By the nondeterministic test assumption we then have $trace(\psi)a = trace(\pi) \in \{trace(\pi') \mid \pi' \in Pref(Outc(\sigma_1^B, \sigma_2^B))\}$.

Direction $\Longleftarrow$

We are given $G_{\mathscr{A}} \sqsubseteq_2 G_{\mathscr{B}}$, and need to prove:

$$out(\mathscr{A}\ after\ \rho) \subseteq out(\mathscr{B}\ after\ \rho) \tag{Q}$$

given an arbitrary trace $\rho \in traces(\mathscr{B})$. According to the deduction tree of Figure 2, which uses abbreviations ϕ_1, ϕ_2, ϕ_3 as defined below, this amounts to proving formula $Q(\rho)$ from $P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A)$, for arbitrary strategies σ_1^A and σ_2^B, and for specifically chosen strategies $\sigma_2^A \in \Sigma_2^{(}G_{\mathscr{A}})$ given ρ, and $\sigma_1^B \in \Sigma_1(G_{\mathscr{B}})$ given ρ, σ_2^A. In this deduction tree $\forall E$ and $\exists E$ refer to the forall elimination and exists elimination rule respectively.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{G_{\mathscr{A}} \sqsubseteq_2 G_{\mathscr{B}} \vdash G_{\mathscr{A}} \sqsubseteq_2 G_{\mathscr{B}}}}{G_{\mathscr{A}} \sqsubseteq_2 G_{\mathscr{B}} \vdash \phi_1(\sigma_2^A)}\,\forall E \qquad \dfrac{\dfrac{\dfrac{}{\phi_2(G_{\mathscr{A}} \sqsubseteq_2 G_{\mathscr{B}}, \phi_2(\sigma_2^A, \sigma_2^B) \vdash \phi_2(\sigma_2^A, \sigma_2^B)}}{G_{\mathscr{A}} \sqsubseteq_2 G_{\mathscr{B}}, \phi_2(\sigma_2^A, \sigma_2^B) \vdash \phi_3(\sigma_2^A, \sigma_2^B, \sigma_1^B)}\,\forall E \qquad \dfrac{}{G_{\mathscr{A}} \sqsubseteq_2 G_{\mathscr{B}}, \phi_2(\sigma_2^A, \sigma_2^B), P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A) \vdash Q(\rho)}\,\text{To Prove}}{G_{\mathscr{A}} \sqsubseteq_2 G_{\mathscr{B}}, \phi_2(\sigma_2^A, \sigma_2^B) \vdash Q(\rho)}\,\exists E}{G_{\mathscr{A}} \sqsubseteq_2 G_{\mathscr{B}} \vdash Q(\rho)}\,\exists E}{G_{\mathscr{A}} \sqsubseteq_2 G_{\mathscr{B}} \vdash \forall \rho \in traces(\mathscr{B}) : Q(\rho)}\,\forall I$$

Figure 2: Deduction tree for case $\Longleftarrow$.

$$\exists \sigma_2^B \in \Sigma_2(G_B), \forall \sigma_1^B \in \Sigma_1(G_B), \exists \sigma_1^A \in \Sigma_1(G_A) : P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A) \tag{ϕ_1}$$

$$\forall \sigma_1^B \in \Sigma_1(G_B), \exists \sigma_1^A \in \Sigma_1(G_A) : P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A) \tag{ϕ_2}$$

$$\exists \sigma_1^A \in \Sigma_1(G_A) : P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A) \tag{ϕ_3}$$

We now will prove the 'To Prove' rule of the deduction tree of Figure 2. We prove by induction on the length of ρ that $\exists \pi \in Pref(Outc(\sigma_1^A, \sigma_2^A)) : trace(\pi) = \rho \wedge \exists \pi_B \in \Pi^{pref}(G_{\mathscr{B}}) : actions(\pi) = actions(\pi_B)$ and $out(\mathscr{A}\ after\ \rho) \subseteq out(\mathscr{B}\ after\ \rho)$.

If $|\rho| = 0$, then $\rho = \varepsilon$. Play prefix $q_0^{\mathscr{A}} \in Pref(Outc(\sigma_1^A, \sigma_2^A))$ has $trace(q_0^{\mathscr{A}}) = \varepsilon = \rho$, and $actions(q_0^{\mathscr{A}}) = actions(q_0^{\mathscr{B}})$. Let $x \in out(\mathscr{A}\ after\ \rho)$. We then have $x \in \Gamma_2^{\mathscr{A}}(q_0^{\mathscr{A}})$ by Definition 8. Because $G_{\mathscr{A}}$ uses the nondeterministic test assumption, and since we can choose $\sigma_2^A(q_0^{\mathscr{A}}) = x$, there is a play prefix $\pi \in Pref(Outc(\sigma_1^A, \sigma_2^A))$ with $trace(\pi) = x$. We have that $P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A)$, so consequently, $x \in \{trace(\pi') \mid \pi' \in Pref(Outc(\sigma_1^B, \sigma_2^B))\}$. Hence, it must be that $x \in \Gamma_2^{\mathscr{B}}(q_0^{\mathscr{B}})$, so by Definition 8 we have $x \in out(\mathscr{B}\ after\ \rho)$.

Suppose that $|\rho| > 0$. Take decomposition $\rho = \rho'\mu$. By the induction hypothesis, we know that $out(\mathscr{A}\ after\ \rho') \subseteq out(\mathscr{B}\ after\ \rho')$, and we have some play prefix $\pi \in Pref(Outc(\sigma_1^A, \sigma_2^A))$ with $trace(\pi) = \rho'$, and a play prefix $\pi_B \in \Pi^{pref}(G_{\mathscr{B}})$ with $actions(\pi) = actions(\pi_B)$.

If $\mu \in L_O^\delta$, then we choose $\sigma_2^A(\pi) = \mu$. By the nondeterministic test assumption, and since $\Gamma_1^{\mathscr{A}}(\pi^q_{|\pi|-1}) \neq \emptyset$ by Definition 8, we then know there is a play prefix π' with $\pi' \in \mathit{Pref}(\mathit{Outc}(\sigma_1^A, \sigma_2^A))$ such that $\mathit{trace}(\pi') = \rho'\mu = \rho$. As $P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A)$ holds, we have a $\pi'_B \in \mathit{Pref}(\mathit{Outc}(\sigma_1^B, \sigma_2^B))\}$ such that $\mathit{trace}(\pi'_B) = \rho$, but to complete the proof, we need a π'_B with $\mathit{actions}(\pi'_B) = \mathit{actions}(\pi')$. If $\sigma_1^A(\pi) = \theta$ then we choose $\sigma_1^B(\pi) = \theta$, as θ is enabled (ρ does not contain stop?, so $\pi^q_{|\pi|-1} \neq (\perp, 1)$). If $\sigma_1^A(\pi) \neq \theta$, we have by the nondeterministic test assumption and $P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A)$ that $\rho'\sigma_1^A(\pi) \in \{\mathit{trace}(\pi'') \mid \pi'' \in \mathit{Pref}(\mathit{Outc}(\sigma_1^B, \sigma_2^B))\}$. Hence, we have a play prefix $\pi'_B \in \mathit{Pref}(\mathit{Outc}(\sigma_1^B, \sigma_2^B))\}$ with $\mathit{actions}(\pi') = \mathit{actions}(\pi'_B)$.

If $\mu \in L_I$ then we check whether $\sigma_1^A(\pi) = \mu$. If so, then by the nondeterministic test assumption we also have a play prefix π' with $\pi' \in \mathit{Pref}(\mathit{Outc}(\sigma_1^A, \sigma_2^A))$ such that $\mathit{trace}(\pi') = \rho'\mu = \rho$. By the nondeterministic test assumption and $P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A)$ that $\rho'\sigma_2^A(\pi) \in \{\mathit{trace}(\pi'') \mid \pi'' \in \mathit{Pref}(\mathit{Outc}(\sigma_1^B, \sigma_2^B))\}$. Hence, we have a play prefix $\pi'_B \in \mathit{Pref}(\mathit{Outc}(\sigma_1^B, \sigma_2^B))\}$ with $\mathit{actions}(\pi') = \mathit{actions}(\pi'_B)$.

If $\sigma_1^A(\pi) \neq \mu$, then we can obtain a contradiction. For all $\pi''_B \in \Pi^{\mathit{pref}}(G_{\mathscr{B}})$ with $\mathit{trace}(\pi''_B) = \rho'$, we choose $\sigma_1^B(\pi''_B) = \mu$. Specifically, $\sigma_1^B(\pi_B) = \mu$. As $\mu \neq \theta$ because $\rho \in L^*$, and since σ_1^A does not cheat on σ_1^B, $\sigma_1^A(\pi) \neq \theta$, so by the nondeterministic test assumption, $\rho'\sigma_1^A(\pi) \in \{\mathit{trace}(\pi'') \mid \pi'' \in \mathit{Pref}(\mathit{Outc}(\sigma_1^A, \sigma_2^A))\}$. However, the set $\{\mathit{trace}(\pi'') \mid \pi''\mathit{Pref}(\mathit{Outc}(\sigma_1^B, \sigma_2^B))$ does not contain a play prefix π'_B with $\mathit{trace}(\pi'_B) = \rho'\sigma_1^A(\pi)$ (because $\sigma_1^B(\pi_B) = \mu$). This contradicts with $P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A)$.

Hence, nomatter whether $\mu \in L_O^\delta$ or $\mu \in L_I$, we obtain a $\pi' \in \mathit{Pref}(\mathit{Outc}(\sigma_1^A, \sigma_2^A))$ with $\mathit{trace}(\pi') = \rho$. Let $x \in \mathit{out}(\mathscr{A}\ \mathit{after}\, \rho)$. By $P(\sigma_2^A, \sigma_2^B, \sigma_1^B, \sigma_1^A)$ and the existence of π' we also have that there exists a $\pi'' \in \mathit{Pref}(\mathit{Outc}(\sigma_1^B, \sigma_2^B))$ with $\mathit{trace}(\pi'') = \rho$. By choosing $\sigma_2^A(\pi') = x$, we obtain $x \in \mathit{out}(\mathscr{B}\ \mathit{after}\, \rho)$, in the same way as for case $|\rho| = 0$.

References

[1] R. Mazala (2001): *Infinite Games*. In: *Automata, Logics, and Infinite Games: A Guide to Current Research*, LNCS 2500, Springer, pp. 23–42, doi:10.1007/3-540-36387-4_2.